# RECONSTRUCTION OF NONSTATIONARY TEMPERATURES FROM THE RESULTS OF MEASUREMENTS ON A HEAT-INSULATED WALL SURFACE

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*The problem of boundary temperature determination from the results of its measurement on a heat-insulated wall surface is analyzed. A solution is based on solving the corresponding direct problem in the form of a Duhamel integral.*

Effective mathematical modeling of heat and mass transfer processes requires a sufficiently exact mathematical description of the boundary conditions – the initial data for calculations. Determination of boundary conditions by physical modeling or a full-scale experiment is expensive and, moreover, inapplicable for the monitoring and regulation of technological parameters. A substantially more exact and cheaper way to find the required boundary conditions is to solve inverse boundary-value problems of heat conduction (IHCP) [1]. Currently, simplicity of programming and adaptation to other geometric forms are also included in the criteria of evaluation of the quality of a method for solving inverse problems [2].

In the present work we suggest a method for solving inverse heat conduction problems in the form of a Duhamel integral that meets the above requirements.

We will consider the problem of determination of the temperature field of a wall from the results of temperature measurements on a heat-insulated surface. In dimensionless variables we write

$$\frac{\partial u}{\partial \mathrm{Fo}} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad \frac{\partial u(0, \mathrm{Fo})}{\partial x} = 0,$$
$$u(x, 0) = 0, \quad u(0, \mathrm{Fo}) = \varphi(\mathrm{Fo}). \tag{1}$$

In fact, the problem reduces to the determination of the law of temperature variation $u(1, \mathrm{Fo}) = f(\mathrm{Fo})$ on the surface $x = 1$. After determining $f(\mathrm{Fo})$ the temperature field may be determined by solving the direct heat conduction problem by one of the known methods [3]. From the solution of the direct problem we find a relationship between the sought function and the experimentally determined function $u(0, \mathrm{Fo})$ in the region of Laplace transforms:

$$u(0, p) = f(p)\frac{1}{\mathrm{ch}\sqrt{p}}, \tag{2}$$

where $u(0, p) = L[u(0, \mathrm{Fo})]$; $f(p) = L[f(\mathrm{Fo})]$; $L$, $p$ are the operator and the parameter of the Laplace transformation; $1/\mathrm{ch}\sqrt{p} = \Pi(p)$ is the transfer function.

Passing to the inverse transforms in equality (2), we obtain a solution of the direct heat conduction problem for an arbitrary change in the boundary temperature in the form of a convolution integral [4]:

$$u(0, \mathrm{Fo}) = \int_0^{\mathrm{Fo}} \Pi(\mathrm{Fo} - \tau) f(\tau)\, d\tau, \tag{3}$$

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where $\Pi(\mathrm{Fo}) = L^{-1}[1/\mathrm{ch}\sqrt{p}\,]]$ is the inverse transform of the transfer function.

Equation (3) represents a Volterra integral equation of the first kind for determination of the unknown function $f(\mathrm{Fo})$ by the experimentally determined dependence $u(0, \mathrm{Fo})$. As is known, such problems are incorrectly stated [5] and for their solution it is necessary to perform regularization.

Since the kernel of integral equation (3) has a complicated form, use of existing algorithms for solving the formulated problem [6, 7] is rather laborious. To simplify the solution obtained, we take advantage of regularization methods that allow a Volterra equation of the first kind to be solved not only in the case of inexact specification of the function $u(0, \mathrm{Fo})$ but also when the kernel of integral equation (3) is known only approximately. Naturally, the accuracy of the solution obtained will depend on the errors in prescribing the kernel and the left-hand side of Eq. (3).

To obtain an approximate kernel that allows simplification of the solution and ensures good accuracy, we use the method of characteristics of imaginary frequencies, consisting in replacing the exact transfer function by an approximate one [8]:

$$1/\mathrm{ch}\,\sqrt{p} \approx \frac{24}{24 + 12p + p^2} = \tilde{\Pi}(p)\,. \tag{4}$$

Substituting the inverse transform of the transfer function into Eq. (3), we regularize the obtained equation by the Lavrent'ev method [9] by transforming it into a Volterra integral equation of the second kind:

$$u(0\,,\ \mathrm{Fo}) = \int\limits_0^{\mathrm{Fo}} \tilde{\Pi}(\mathrm{Fo} - \tau)\, f(\tau)\, d\tau + \alpha f(\mathrm{Fo})\,, \tag{5}$$

where $\alpha > 0$ is the regularization parameter. Performing the Laplace transformation of Eq. (5) we arrive at

$$f(p) = u(0\,,\ p)/(\tilde{\Pi}(M\,,\ p) + \alpha)\,. \tag{6}$$

Passing, in Eq. (6), to the inverse transforms, we obtain the working formula

$$u(0\,,\ \mathrm{Fo}) = \frac{1}{\alpha}\left[u(0\,,\ \mathrm{Fo}) - k\int\limits_0^{\mathrm{Fo}} \mathrm{e}^{-\beta(\mathrm{Fo}-\tau)} \sin\left[\omega(\mathrm{Fo} - \tau)\right] u(0\,,\ \tau)\, d\tau\right], \tag{7}$$

where $-\beta = 6$; $\omega = \sqrt{((1 + \alpha)/\alpha)24 - \beta^2}$; $k = 24(\alpha\omega)$.

We now analyze formula (7) from the viewpoint of Tikhonov regularization for equations of convolution type without restrictions, unlike the Lavrent'ev method, on the form of the kernel of the integral equation. In accordance with this method, regularization of the problem is accomplished by introducing the stabilizing multiplier $s(p, \alpha)$, which leads to replacement of the "unstable solution" of Eq. (3) in the transform region by the expression

$$u(0\,,\ p)\frac{1}{\Pi(p)} \to u(0\,,\ p)\frac{s(p\,,\ \alpha)}{\Pi(p)}\,. \tag{8}$$

The stabilizing multiplier satisfies a number of conditions [5] that provide for the existence of the inverse transform of $s(p, \alpha)/\Pi(p)$ and passage of the obtained solution into the exact one as $\alpha \to 0$. Equation (6) obtained above corresponds to a choice of the multiplier $s(p, \alpha)$ in the form

$$\tilde{s}(p\,,\ \alpha) = \frac{\Pi(M\,,\ p)}{(\tilde{\Pi}(M\,,\ p) + \alpha)}\,. \tag{9}$$

With an appropriate choice of $\tilde{\Pi}(p)$ the multiplier $\tilde{s}(p, \alpha)$ satisfies all the conditions of the stabilizing multiplier with the exception of the condition $s(p, 0) \equiv 1$. Fulfillment of this condition ensures the coincidence of the obtained solution with the exact one at $\alpha = 0$, which corresponds to exact initial data. For $\tilde{s}(p, \alpha)$ this condition is replaced by the weaker condition $\tilde{s}(p,0) \to 1$ as $p \to 0$. Consequently, with use of multiplier (9) the solution of the problem

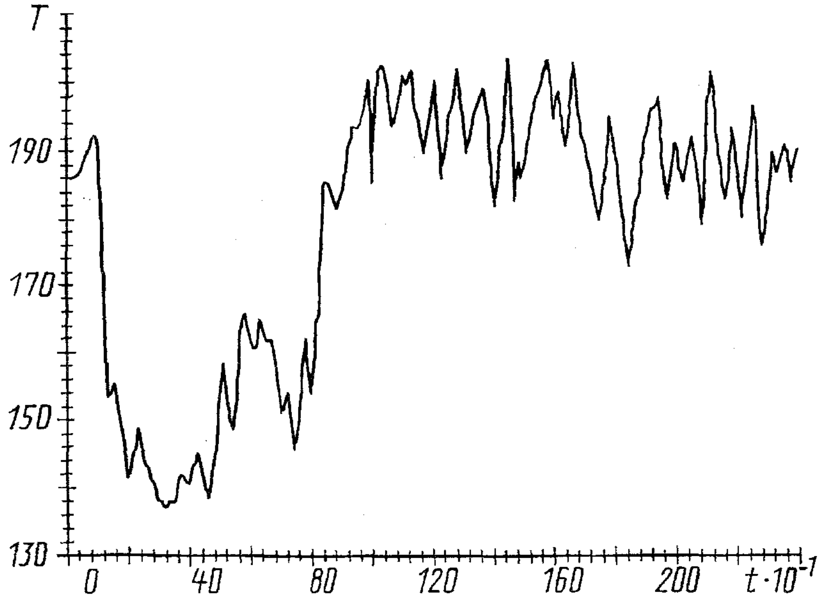

Fig. 1. Graph of the initial boundary temperature. $T$, $^{\circ}$C; $t$, sec.

based on faithful initial data passes into the exact one for large moments of time. However, in the cases of interest in practice the left-hand side of Eq. (3) is known inexactly and the regularization parameter is $\alpha \neq 0$. In this case, the stabilizing factor is $0 \le s(p, \alpha) \le 1$ for all values of $p$ and $\alpha$. Correspondingly, the multiplier $\tilde{s}(p, \alpha)$ may be considered as the stabilizing one for inexactly prescribed initial data if the function $\tilde{\Pi}(p)$ is chosen so that $0 \le \tilde{s}(p, \alpha) \le 1$. In particular, this condition is fulfilled when an approximation is chosen in the form of (4). In some cases, the accuracy of prescribing the approximate transfer function $\tilde{\Pi}(p)$ may be improved by increasing the number of retained terms in the power expansion of the function $1/\Pi(p)$. Use of the described procedure has shown that approximation (4) is sufficient when the initial data for solving the inverse problem are obtained by solving numerically the corresponding direct heat conduction problem.

To perform calculations by formula (7), it is necessary to determine the regularization parameter $\alpha$. The optimum value of $\alpha$ depends on the errors in the experimental data and the accuracy of prescribing the kernel of Eq. (5). Since in practice the measurement errors are, as a rule, unknown, we used the following criterion for choosing a choice of the regularization parameter [10]:

$$\| y_{\alpha_{i+1}} - y_{\alpha_i} \| = \min\,, \tag{10}$$

where $\alpha_{i+1} = \theta\alpha_i$, $0 < \theta < 1$.

The choice of this criterion makes it possible to determine the regularization parameter for the given measuring complex in a separate experiment or in the stage of trial starting of the measuring setup. In further calculations, $\alpha$ may be considered to be fixed and equal to the determined value. It is pertinent to note that criterion (10) for finding the quasioptimum regularization parameter is substantiated only for some special classes of inverse problems. However its use for solving various IHCPs, using the procedure considered, yields good results. Here, the range of $\alpha$ values yielding close results appears to be fairly wide.

The suggested procedure allows boundary-value IHCPs to be solved efficiently for bodies of simple geometry. Thus, a generalization of the above problem is the temperature determination at the boundary $x = 1$ by using measurement data for the temperature $u(0, \text{Fo})$ and the heat flux at the surface $x = 0$. We write the solution of the corresponding direct problem in the Laplace transform region in the form

$$u\,(0\,,\ p) + q\,(p)\ \text{th}\,\sqrt{p}/\sqrt{p} = u\,(1\,,\ p)/\text{ch}\,\sqrt{p}\,. \tag{11}$$

Since the dependences $u(0, \text{Fo})$ and $q(\text{Fo})$ are conditionally prescribed, determination of the inverse transform of $\text{th}\sqrt{p}/\sqrt{p}$ makes it possible to determine the inverse transform of the left-hand side of Eq. (11):

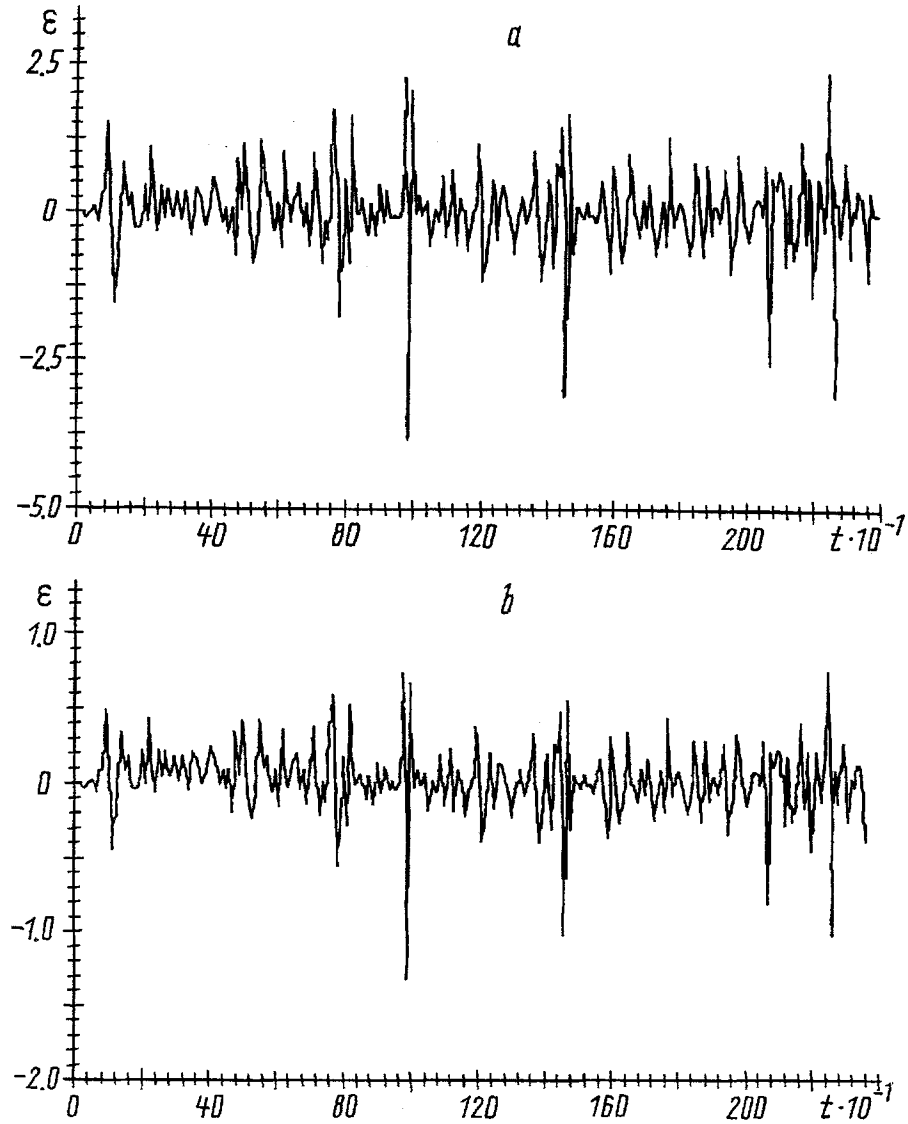


Fig. 2. Errors in the reconstruction of boundary (a) and integral mean temperatures (b). $\varepsilon$, %.

$$\varphi(\mathrm{Fo}) = L^{-1}\,[u(0,\,p) + q(p)\,\mathrm{th}\,\sqrt{p}/\sqrt{p}\,]. \tag{12}$$

Next, considering $\varphi$(Fo) to be the new prescribed function and passing to the inverse transforms in Eq. (11), we obtain an integral equation that coincides with Eq. (3). Therefore, the present problem coincides with the previous one after preliminary recalculation of the initial data.

In engineering calculations, use is often made of an integral mean temperature. It may be determined directly from the results of temperature measurements on a heat-insulated surface. In the region of Laplace transforms the transforms of the integral mean temperature and the temperature on a heat-insulated surface are related as [3]:

$$u(0,\,p) = u_{\mathrm{int.m}}(p)\,\sqrt{p}/\mathrm{sh}\,\sqrt{p}\,. \tag{13}$$

Substituting the approximate function [8] for the transfer function in Eq. (3) and performing regularization similarly to the above, we again arrive at Eq. (7) but with the other values of the coefficients in it:

$$\beta = 10\,,\quad \omega = \sqrt{\left(\frac{1+\alpha}{\alpha}\,120 - \beta^2\right)}\,,\quad k = 120/(\alpha\omega)\,.$$

Solving analogous problems for a cylinder and a sphere also yields Eq. (7), in which the coefficients are determined from approximation of the corresponding transfer function.

Thus, many inverse heat conduction problems for bodies of simple geometry may be solved on a unified basis using formula (7). Here, it is necessary to recalculate preliminarily the initial data for some problems.

We now illustrate reconstruction of a boundary temperature from results of its measurement on a heat-insulated surface. As the initial boundary temperature, we took experimental data for a 6.5-mm-thick wall with a thermal diffusivity of $0.445 \cdot 10^{-5}$ m$^2$/sec and an observation interval of 10 sec, which corresponds to an observation step with respect to the Fourier number equal to 1.053 (see Fig. 1). The initial data for solving the inverse problem were obtained by solving numerically the direct problem and rounding off the obtained results to tenths of a degree. The obtained solution of the inverse problem was compared to the prescribed law of boundary temperature variation. The relative error of reconstruction of the boundary dependence is shown in Fig. 2a.

As an analysis of the obtained graphs shows, at the majority of points the error does not exceed 1%. The maximum value of the relative error is 3.8%, which corresponds to large temperature fluctuations for a short time. Since surface temperature fluctuations are smoothed at internal points, we may assert that the errors in the temperature field determination do not exceed the corresponding errors in the determination of the boundary conditions.

Figure 2b shows a graph of reconstruction of an integral mean temperature. For comparison, we used data of numerical calculations. A comparison of the errors of reconstruction of the boundary and integral mean temperatures shows that the latter are approximately 2.5-fold smaller and, in fact, do not exceed 0.5% at all the points.

The simplicity of calculation formula (7) allows development of a high-speed package of programs that requires a small memory and is applicable for solution of the boundary-value IHTPs for bodies of canonical form. In practice, calculations have demonstrated that the suggested procedure is applicable for both large and small steps with respect to the Fourier number. The results may be processed on an IBM-XT computer in real time with a measurement step with respect to the Fourier number equal to approximately 0.1. This allows the algorithm to be used in systems for controlling technological parameters that are inaccessible to direct measurements.

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